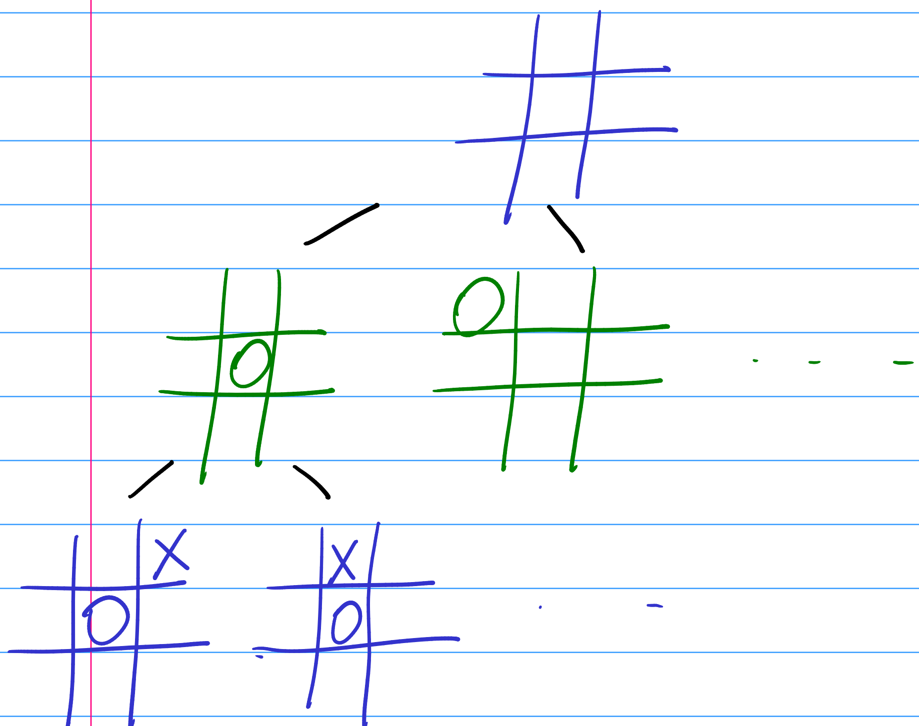
Topic 6 : Game Theory

SG Function , Search

Game Tree



Current Player

Alice (O)

Bob (X)

Alice (O)

Losing State: Current player loses, no matter what he does.

(Predefined losing state: being checkmated, etc.)

Winning State: Current player wins if he finds the move

Search on the Game Tree

Theorem: If S leads to a losing state. S is winning.
If S only leads to winning states. S is losing.

Endgame

Theorem: If S leads to a losing state, S is winning.
If S only leads to winning states, S is losing.

Endgame

Theorem: (1) If S leads to a losing state. S is winning.
(2) If S *only* leads to winning states. S is losing.

Reverse Topo Search

1) Start with predefined losing states
2) Recursively apply Thm. 1 and 2

   If $S \to S'$ and S' is losing, then $\text{move}(S) \leftarrow \text{move}(S')+1$.

   If all outgoing edges of S are winning, then $\text{move}(S) \leftarrow \max_{S'} (\text{move}(S'))$

   Reverse BFS (v updates every v' s.t. $v' \to v$)

3) All vertices not visited are neither winning or losing = drawing

Nim

Several pile of stones, e.g {1, 2, 3}

Every move a player take any non-zero amount from one pile

Whoever takes the last stone wins.

Impartial — Both players share the moveset

Solution: XOR every pile together, e.g. $1 \oplus 2 \oplus 3 = 0$

(known as the SG number of the game)

If it's 0, second player wins. (losing state)

Otherwise, first player wins. (winning state)

SG Theorem

(1) A game of one pile of n has SG number n.

(2) When we 'combine' two Nim Games together.

we get a new game with SG number $a \oplus b$.

☆: A Nim game of SG n is functionally equivalent to a pile of n stones under combination.

SG Thm: Any impartial game is equivalent to a Nim game (a pile of n stones)

We usually just call the SG number of the game to be n.

Analysis of SG Numbers

Pre-made Tables (see reference ...)

mex function:

$$\text{mex}(X) = \min\{x \mid x \in N \wedge x \notin X\}$$

e.g. $\text{mex}(\{0,1,3\}) = 2$.

If a state S leads to $S_1, S_2 \cdots S_n$.

Then $SG(S) = \text{mex}(\{SG(S_1) \cdots SG(S_n)\})$.

e.g. A pile of n stones has SG n

since it leads to $(n-1)$ stones, ..., 0 stones

Vim Or Not Vim?

Sequential Vim

Vim